A DESCRIPTION OF A CLASS OF FINITE SEMIGROUPS THAT ARE NEAR TO BEING MALCEV NILPOTENT

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Abstract. In this paper we continue the investigations on the algebraic structure of a finite semigroup S that is determined by its associated upper non-nilpotent graph $\mathcal{N}_S$. The vertices of this graph are the elements of S and two vertices are adjacent if they generate a semigroup that is not nilpotent (in the sense of Malcev). We introduce a class of semigroups in which the Mal'cev nilpotent property lifts through ideal chains. We call this the class of pseudo-nilpotent semigroups. The definition is such that the global information that a semigroup is not nilpotent induces local information, i.e. some two-generated subsemigroups are not nilpotent. It turns out that a finite monoid (in particular, a finite group) is pseudo-nilpotent if and only if it is nilpotent. Our main result is a description of pseudo-nilpotent finite semigroups S in terms of their associated graph $\mathcal{N}_S$. In particular, S has a largest nilpotent ideal, say K, and S/K is a 0-disjoint union of its connected components (adjoined with a zero) with at least two elements.

1. Introduction

For a semigroup S with elements $x, y, z_1, z_2, \dots$ one recursively defines two sequences

$$\lambda_n = \lambda_n(x, y, z_1, \dots, z_n) \quad \text{and} \quad \rho_n = \rho_n(x, y, z_1, \dots, z_n)$$

by

$$\lambda_0 = x, \quad \rho_0 = y$$

and

$$\lambda_{n+1} = \lambda_n z_{n+1} \rho_n, \quad \rho_{n+1} = \rho_n z_{n+1} \lambda_n.$$

A semigroup is said to be *nilpotent* (in the sense of Mal'cev [8], denote (MN) in [6]) if there exists a positive integer n such that

$$\lambda_n(a, b, c_1, \dots, c_n) = \rho_n(a, b, c_1, \dots, c_n)$$

for all a, b in S and $c_1, \dots, c_n$ in S^1. The smallest such n is called the nilpotency class of S. It is well known that a group G is nilpotent of class n if and only if it is nilpotent of class n in the classical sense. Nilpotent semigroups and their semigroup algebras have been investigated in [4]. For

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example, it is proved that a completely 0-simple semigroup S is nilpotent if and only if S is an inverse semigroup with nilpotent maximal subgroups. If S is a semigroup with a zero θ, then obviously an ideal I with $I^n = \{\theta\}$ is a nilpotent semigroup as well. Several consequences of the (Mal'cev) nilpotence have appeared in the literature. For example, in [9] a semigroup S is said to be *Neumann-Taylor* (NT) if there exists a positive integer $n \geq 2$ such that

$$\lambda_n(a, b, 1, c_2, \dots, c_n) = \rho_n(a, b, 1, c_2, \dots, c_n)$$

for all $a, b, c_2, \dots, c_n$ in S^1. A semigroup S is said to be *positively Engel* (PE) if for some positive integer $n \geq 2$,

$$\lambda_n(a, b, 1, 1, c, c^2, \dots, c^{n-2}) = \rho_n(a, b, 1, 1, c, c^2, \dots, c^{n-2})$$

for all a, b in S and $c \in S^1$. A semigroup S is said to be *weakly Mal'cev nilpotent* (WMN) if for some positive integer n,

$$\lambda_n(a, b, c_1, c_2, c_1, c_2, \dots) = \rho_n(a, b, c_1, c_2, c_1, c_2, \dots)$$

for all a, b in S and $c_1, c_2 \in S^1$. These classes of semigroups have been studied in [6].

In [7] we initiated the investigations on the upper non-nilpotent graph $\mathcal{N}_S$ of a finite semigroup S. Recall that the vertices of $\mathcal{N}_S$ are the elements of S and there is an edge between x and y if the semigroup generated by x and y, denoted by $\langle x, y\rangle$, is not nilpotent. Note that $\mathcal{N}_S$ is empty if S is a nilpotent semigroup. We state some of the results obtained. If a finite semigroup S has empty upper non-nilpotent graph then S is positively Engel. On the other hand, a semigroup has a complete upper non-nilpotent graph if and only if it is a completely simple semigroup that is a band. The main result says that if all connected $\mathcal{N}_S$-components of a semigroup S are complete (with at least two elements) then S is a band that is a semilattice of its connected components and, moreover, S is an iterated total ideal extension of its connected components. Further, it is shown that some graphs, such as a cycle C_n on n vertices (with $n \geq 5$), are not the upper non-nilpotent graph of a semigroup. Also, there is precisely one graph on 4 vertices that is not the upper non-nilpotent graph of a semigroup with 4 elements.

In this paper we continue these investigations. We introduce a class of semigroups in which the Mal'cev nilpotent property lifts through ideal chains. We call this the class of pseudo-nilpotent semigroups. It turns out that a finite monoid (in particular, a finite group) is pseudo-nilpotent if and only if it is nilpotent. Our main result is a description of pseudo-nilpotent finite semigroup S in terms of their associated graph $\mathcal{N}_S$. In particular, S has a largest nilpotent ideal, say K, and S/K is a 0-disjoint union of its connected components (adjoined with a zero) with at least two elements.

For standard notations and terminology we refer to [3].

2. PSEUDO-NILPOTENT SEMIGROUPS

Suppose that I is an ideal of a semigroup S. If S is nilpotent then clearly so are the semigroups I and S/I. However in general the converse fails (see Example 2.3 in [4]). We will introduce a class of semigroups, called the pseudo-nilpotent semigroups, for which the converse does hold. The definition is motivated by the following lemma proved in [7].

Lemma 2.1. *A finite semigroup S is not nilpotent if and only if there exists a positive integer m, distinct elements $x, y \in S$ and $w_1, w_2, \ldots, w_m \in S^1$, such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$, $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$ (note that for the converse one does not need that S is finite).*

For convenience we call the empty set an ideal of S and thus, by definition, $S/\varnothing = S$.

Definition 2.2. *A semigroup S is said to be pseudo-nilpotent if*

$$\lambda_t(x, y, w_1, \ldots, w_t) \neq \rho_t(x, y, w_1, \ldots, w_t),$$

$$(\lambda_t(x, y, w_1, \ldots, w_t), \rho_t(x, y, w_1, \ldots, w_t)) =$$

$$(\lambda_m(x, y, w_1, \ldots, w_m), \rho_m(x, y, w_1, \ldots, w_m)),$$

for $x, y \in S$, $w_1, \ldots, w_m \in S^1$, I an ideal (possibly empty) of $\langle x, y, w_1, \ldots, w_m\rangle$, $\lambda_m, \rho_m \notin I$, and non-negative integers $t < m$, implies that there are edges in $\mathcal{N}_{\langle x,y,w_1,\ldots,w_m\rangle/I}$ between λ_i and ρ_i for every $0 \le i \le m$.

So in some sense the global condition that S is not nilpotent determines local information, i.e. some two-generated subsemigroups are not nilpotent.

Examples of pseudo-nilpotent semigroups are nilpotent semigroups and semigroups with complete upper non-nilpotent graphs. Obviously, subsemigroups and Rees factor semigroups of pseudo-nilpotent semigroups are also pseudo-nilpotent. Hence, the pseudo-nilpotent condition yields restrictions on the completely 0-simple components of a semigroup. It easily can be verified that the completely 0-simple semigroup $\mathcal{M}^0(\{e\}, 2, 2; \begin{pmatrix} 1 & 1 \\ \theta & 1 \end{pmatrix})$ is not pseudo-nilpotent.

Note that if in the definition of pseudo-nilpotent semigroup S one would assume that I is an ideal of S and require that there are edges $\mathcal{N}_{S/I}$ between λ_i and ρ_i for every $0 \le i \le m$ then subsemigroups or Rees factors of pseudo-nilpotent semigroups need not to inherit this condition. Hence the requirement that I is an ideal of $\langle x, y, w_1, \ldots, w_m\rangle$

Lemma 2.3. *Let S be a pseudo-nilpotent finite semigroup. If I is an ideal of S such that I and S/I are nilpotent, then S is nilpotent.*

Proof. Suppose that S is not nilpotent. Then by Lemma 2.1, there exists a positive integer m, distinct elements $x, y \in S$ and elements $w_1, w_2, \ldots, w_m \in S^1$, such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$ and $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$. As S is pseudo-nilpotent, this implies that there is an edge in $\mathcal{N}_S$ between

x and y, i.e. $\langle x,y\rangle$ is not nilpotent. If both x and y are not in I, then none of the elements $w_1,w_2,\dots,w_m,\lambda_i(x,y,w_1,\dots,w_i),\rho_i(x,y,w_1,\dots,w_i)$ (for $1\le i\le m$) belong to I, as I is an ideal of S. However, this is in contradiction with S/I being nilpotent. So $x\in I$ or $y\in I$.

The equalities $x=\lambda_m(x,y,w_1,w_2,\dots,w_m)$ and $y=\rho_m(x,y,w_1,w_2,\dots,w_m)$ imply that there exist elements $a,b,a',b'\in S^1$ such that $x=ayb, y=a'xb'$. Again because I is an ideal of S, it follows that $x,y\in I$. Since $\langle x,y\rangle$ is not nilpotent, this yields a contradiction with I being nilpotent. $\square$

The pseudo-nilpotent condition also includes restrictions on how elements in different principal factors multiply. Indeed, for if an ideal I of a semigroup S and its factor semigroup S/I are pseudo-nilpotent, then it does not necessarily follow that S is also pseudo-nilpotent. For example, the semigroup $S=\{a,b,c,d,e\}$ with multiplication table[1]

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	e
c	a	a	c	d	a
d	d	d	d	d	d
e	e	e	e	e	e

has an ideal $I=\{a,d,e\}$ with I and S/I pseudo-nilpotent. But S is not pseudo-nilpotent, because there is no edge in $\mathcal{N}_S$ between b and e but

$$(\lambda_1(b,e,a),\rho_1(b,e,a))=(\lambda_2(b,e,a,a),\rho_2(b,e,a,a)).$$

In [7] an example is given of a finite semigroup that is not nilpotent and has empty upper non-nilpotent graph. The next lemma shows that such an example can not be pseudo-nilpotent.

Lemma 2.4. *If a finite semigroup S is pseudo-nilpotent and $\mathcal{N}_S$ is empty, then S is nilpotent.*

Proof. Indeed, suppose the contrary. That is, assume S is not nilpotent. Then, by Lemma 2.1, there exists a positive integer m, distinct elements $x,y\in S$ and elements $w_1,w_2,\dots,w_m\in S^1$, such that $x=\lambda_m(x,y,w_1,w_2,\dots,w_m)$ and $y=\rho_m(x,y,w_1,w_2,\dots,w_m)$. However, as S is pseudo-nilpotent, we get that $\langle x,y\rangle$ is not nilpotent. This contradicts with $\mathcal{N}_S=\varnothing$. $\square$

Clearly for a finite semigroup we have the following implications

$$(MN)\Rightarrow(WMN),\ (MN)\Rightarrow(NT)\ \text{ and } (MN)\Rightarrow\text{pseudo-nilpotent}.$$

In [7] an example is given of a finite semigroup S which is (NT) but not (MN) and with $\mathcal{N}_S=\varnothing$. So S is not pseudo-nilpotent. Of course a finite semigroup S for which $\mathcal{N}_S$ is complete and $\mathcal{N}_S\neq\varnothing$ is pseudo-nilpotent but it is neither (NT) nor (WMN). Recall from Proposition 3.4 in [7] that every

[1] In order to check the associativity law for the constructed examples, a software is developed in C++ programming language.

finite semigroup S for which $\mathcal{N}_S$ is complete and $|S| > 1$, is isomorphic with a completely simple semigroup $\mathcal{M}(\{e\}, I, \Lambda; P)$ (in particular it is a band). As all elements of P are equal to e, we have, for all $a, b \in S$,

$$a = \lambda_2(a, b, 1_S, 1_S), b = \rho_2(a, b, 1_S, 1_S).$$

It follows that such semigroup is neither (NT) nor (WMN).

Hence the classes (WMN), (NT) and pseudo-nilpotent are pairwise distinct classes containing the Mal'cev nilpotent semigroups (Recall from Corollary 12 in [6] that a finite semigroup is (WMN) if and only if it is (MN).). However, in the following lemma we show that for finite groups these notions are the same.

Lemma 2.5. *A finite group is pseudo-nilpotent if and only if it is nilpotent.*

Proof. Let G be a finite group. Of course if G is nilpotent then G is pseudo-nilpotent. For the converse, recall that G is nilpotent if and only if it is Neumann Taylor. So assume G is pseudo-nilpotent. We need to prove that G is Neumann Taylor. We prove this by contradiction. So suppose G is not (NT). Let $k = |G|$. There exist distinct elements $a, b \in G$ and $w_1, \ldots, w_{k^2} \in G$ such that

$$\lambda_{k^2+1}(a, b, 1_G, w_1, \ldots, w_{k^2}) \neq \rho_{k^2+1}(a, b, 1_G, w_1, \ldots, w_{k^2}).$$

Since $|G \times G| = k^2$ there exist non-negative integers $t, r \leq k^2$, $t < r$ with

$$\begin{aligned} &(\lambda_{t+1}(a, b, 1_G, w_1, \ldots, w_t), \rho_{t+1}(a, b, 1_G, w_1, \ldots, w_t)) \\ &= \quad (\lambda_{r+1}(a, b, 1_G, w_1, \ldots, w_r), \rho_{r+1}(a, b, 1_G, w_1, \ldots, w_r)). \end{aligned}$$

Therefore we also have

$$\begin{aligned} &(\lambda_{t+1}(a, 1_G, b, w_1, \ldots, w_t), \rho_{t+1}(a, 1_G, b, w_1, \ldots, w_t)) \\ &= \quad (\lambda_{r+1}(a, 1_G, b, w_1, \ldots, w_r), \rho_{r+1}(a, 1_G, b, w_1, \ldots, w_r)). \end{aligned}$$

Because G is pseudo-nilpotent, it implies that there is an edge in $\mathcal{N}_S$ between 1_G and a, a contradiction. $\square$

Recall from [7] that the lower non-nilpotent graph $\mathcal{L}_S$ of a semigroup S is the graph whose vertices are the elements of S and there is an edge between two distinct vertices $x, y \in S$ if and only if there exist elements $w_1, w_2, \ldots, w_n$ in $\langle x, y\rangle^1$ with $x = \lambda_n(x, y, w_1, w_2, \ldots, w_n)$ and $y = \rho_n(x, y, w_1, w_2, \ldots, w_n)$. Clearly, because of Lemma 2.1, $\mathcal{L}_S$ is a subgraph of $\mathcal{N}_S$. In general $\mathcal{L}_S$ and $\mathcal{N}_S$ are different.

Proposition 2.6. *Let S be a finite semigroup. If S is pseudo-nilpotent and $\mathcal{L}_S$ is empty, then $\mathcal{N}_S$ is empty.*

Proof. Suppose $\mathcal{N}_S$ is not empty. Then S is not nilpotent and hence, by Lemma 2.1, there exists a positive integer m and distinct elements $x, y \in S$ and $w_1, w_2, \ldots, w_m \in S^1$ such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$ and $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$. Since, by assumption, S is pseudo-nilpotent, we get that $S_1 = \langle x, y\rangle$ is not nilpotent.

Note that the subsemigroup S_1 is pseudo-nilpotent. Since S_1 is not nilpotent, there exists a positive integer m_1, distinct elements $x_1, y_1 \in S_1$ and $w_1', w_2', \ldots, w_{m_1}' \in S_1^1$ such that $x_1 = \lambda_{m_1}(x_1, y_1, w_1', w_2', \ldots, w_{m_1}')$ and $y_1 = \rho_{m_1}(x_1, y_1, w_1', w_2', \ldots, w_{m_1}')$. Therefore $S_2 = \langle x_1, y_1 \rangle$ is not nilpotent.

Let $x_0 = x$ and $y_0 = y$. By induction we obtain for each non-negative integer i a subsemigroup S_i of S such that

$$S_0 = S, \quad S_i = \langle x_{i-1}, y_{i-1} \rangle (\text{ for } i \geq 1)$$

and elements

$$x_i, y_i, w_1^{(i)}, w_2^{(i)}, \ldots, w_{m_i}^{(i)} \in S_i^1$$

such that

$$x_i = \lambda_{m_i}(x_i, y_i, w_1^{(i)}, \ldots, w_{m_i}^{(i)}), y_i = \rho_{m_i}(x_i, y_i, w_1^{(i)}, \ldots, w_{m_i}^{(i)})$$

and

$$x_i \neq y_i.$$

Clearly,

$$S \supseteq S_1 \supseteq S_2 \supseteq \cdots$$

of subsemigroups of S. Since S is finite, we get that $S_t = S_{t+1}$ for some positive integer t. Hence $x_t \neq y_t$ and

$$x_t = \lambda_{m_t}(x_t, y_t, w_1^{(t)}, \ldots, w_{m_t}^{(t)}), \ y_t = \rho_{m_t}(x_t, y_t, w_1^{(t)}, \ldots, w_{m_t}^{(t)}),$$

$$x_t, y_t, w_1^{(t)}, w_2^{(t)}, \ldots, w_{m_t}^{(t)} \in S_t^1 = S_{t+1}^1 = \langle x_t, y_t \rangle.$$

So, in the graph $\mathcal{L}_{\langle x_t, y_t \rangle}$ there is an edge between x_t and y_t. Since $\mathcal{L}_{\langle x_t, y_t \rangle}$ is a subgraph of $\mathcal{L}_S$, we obtain that the graph $\mathcal{L}_S$ is not empty, as desired. □

We finish this section with proving one more restriction that the pseudo-nilpotent condition imposes.

Lemma 2.7. *Let S be a pseudo-nilpotent finite semigroup. If I is an ideal of S and $\mathcal{N}_I$ is empty, then elements of I are isolated vertices in $\mathcal{N}_S$.*

Proof. We prove the result by contradiction. So, suppose that there exists an edge in $\mathcal{N}_S$ between $a \in S \backslash I$ and $b \in I$. Hence $\langle a, b \rangle = \langle a \rangle \cup B$, for some subset B of I. Because $\langle a, b \rangle$ is not nilpotent, by Lemma 2.1, there exists a positive integer m and elements $x, y, w_1, w_2, \ldots, w_m$ of $\langle a, b \rangle^1$ such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m), y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$ and $x \neq y$. As the cyclic semigroup $\langle a \rangle$ is nilpotent, we get that $\{x, y, w_1, w_2, \ldots, w_m\} \cap B \neq \varnothing$. Because I is an ideal, the equalities $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$ and $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$ yield that $x, y \in I$. Since, by assumption, S is pseudo-nilpotent, we know that $\langle x, y \rangle$ is not nilpotent, i.e. there is an edge in $\mathcal{N}_S$ between x and y. However, there is contradiction with $\mathcal{N}_I = \varnothing$. □

3. A description of pseudo-nilpotent semigroups

We begin by describing the completely 0-simple semigroups that are pseudo-nilpotent. The non-zero elements of a completely 0-simple semigroup $\mathcal{M}^0(G, I, \Lambda; P)$ over a group G are denoted as $(g; i, \lambda)$, with $g \in G$, $i \in I$ and $\lambda \in \Lambda$. Also, we denote the sets $\{(g; i, \lambda) \mid g \in G\}, \{(g; i, \lambda) \mid g \in G, \lambda \in \Lambda\}$ and $\{(g; i, \lambda) \mid g \in G, i \in I\}$ by $G_{i,\lambda}, G_{i,\star}$ and $G_{\star,\lambda}$ respectively.

Lemma 3.1. *Let S be a pseudo-nilpotent finite semigroup. Assume A and B are ideals of S such that $A \subseteq B$ and $B/A = \mathcal{M}^0(G, I, \Lambda; P)$ is a regular Rees matrix semigroup. Then either B/A is nilpotent (i.e. G is nilpotent and B/A is an inverse semigroup) or $|I| + |\Lambda| > 2$, G is a nilpotent group and all entries of P are non-zero (i.e. B/A is a union of groups, and thus $B \backslash A$ is a semigroup).*

Conversely, any completely 0-simple semigroup $\mathcal{M}^0(G, I, \Lambda; P)$, with G a nilpotent group and all entries of P non-zero, is pseudo-nilpotent. Furthermore, if $|I| + |\Lambda| > 2$ and $(i, \lambda) \neq (i', \lambda')$ then $\langle (g; i, \lambda), (g'; i', \lambda') \rangle$ is not nilpotent for all $g, g' \in G$. In particular, the upper non-nilpotent graph of $\mathcal{M}^0(G, I, \Lambda; P) \backslash \{\theta\}$ is connected and regular.

Proof. Because B/A is regular, each row and column of P contains a non-zero entry. Lemma 2.1 of [4] implies that if each row and column does not contain more than one non-zero element, then B/A is nilpotent.

Suppose B/A is not nilpotent. Then, without loss of generality, we may assume that some row of P contains more than one non-zero element. Say $p_{j,i_1}, p_{j,i_2} \neq \theta$. If $\Lambda = \{j\}$, because B/A is regular, all columns are non-zero, and hence all elements of P are non-zero. Otherwise, let $j' \in \Lambda$, $j' \neq j$. Because B/A is regular, there exists $\alpha \in I$ such that $p_{j',\alpha} \neq \theta$. Now we have

$$((p_{j,i_1}^{-1}; i_1, j'), (p_{j,i_2}^{-1}; i_2, j')) = (\lambda_1((p_{j,i_1}^{-1}; i_1, j'), (p_{j,i_2}^{-1}; i_2, j'), (p_{j',\alpha}^{-1}; \alpha, j)),$$
$$\rho_1((p_{j,i_1}^{-1}; i_1, j'), (p_{j,i_2}^{-1}; i_2, j'), (p_{j',\alpha}^{-1}; \alpha, j))).$$

Hence, because S is pseudo-nilpotent, there is an edge in $\mathcal{N}_{S/A}$ between $(p_{j,i_1}^{-1}; i_1, j')$ and $(p_{j,i_2}^{-1}; i_2, j')$. Consequently, by Lemma 2.1, there exists a positive integer m and elements $x, y, w_1, w_2, \ldots, w_m$ of $\langle (p_{j,i_1}^{-1}; i_1, j'), (p_{j,i_2}^{-1}; i_2, j') \rangle^1$ such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$, $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$ and $x \neq y$.

If $p_{j',i_1} = \theta$ then, for $g \in G$ and working in S/A,

$$\langle (p_{j,i_1}^{-1}; i_1, j'), (p_{j,i_2}^{-1}; i_2, j') \rangle (g; i_1, j') = \theta.$$

Hence $\{x, y, w_1, w_2, \ldots, w_m\} \subseteq G_{i_2,j'}$ and $p_{j',i_2} \neq \theta$, because it is impossible that $\theta \in \{x, y\}\{w_1, w_2, \ldots, w_m\}$ or $\theta \in \{w_1, w_2, \ldots, w_m\}\{x, y\}$.

Consequently, the group $G_{i_2,j'} \cong G$ is not nilpotent. However, as a subsemigroup of S, it is pseudo-nilpotent and thus, by Lemma 2.5, it is a nilpotent group. This yields a contradiction. So we have proved that $p_{j',i_1} \neq \theta$. Similarly $p_{j',i_2} \neq \theta$. Thus, $p_{k,i_1} \neq \theta, p_{k,i_2} \neq \theta$ for all $k \in \Lambda$. That is, columns i_1 and i_2 of P do not contain θ.

Let $i_3 \in I$. Because B/A is regular, there exists $l \in \Lambda$ such that $p_{l,i_3} \neq \theta$. As also $p_{l,i_1} \neq \theta$, the above yields that column i_3 of P also not contain θ. So all entries of P are different from θ. This finishes the first part of the result.

The second part easily can be verified. □

Next we describe how a completely 0-simple factor fits into a pseudo-nilpotent semigroup, in particular, we investigate the restrictions on how elements in different principal factors multiply. To do so we introduce some notations. Let S be a semigroup with a zero θ. For an ideal J of S let

$$F_J(S) = F_J = \{s \in S\backslash J \mid sj \neq \theta \text{ or } js \neq \theta \text{ for some } j \in J\}$$

and let

$$F'_J(S) = F'_J = S\backslash(F_J \cup (J\backslash\{\theta\})).$$

Obviously, $F'_J \cup J$ is an ideal of S and if $x \in F'_J$ and $y \in J$, then $\langle x, y\rangle$ is nilpotent, i.e. there is no edge in $\mathcal{N}_S$ between x and y.

Note however that for an arbitrary pseudo-nilpotent semigroup S with zero the set $F'_J(S)$ is not necessarily an ideal for an ideal J of S. For example let $S = \{\theta, a, b, c\}$ be the semigroup with Cayley table

	θ	a	b	c
θ	θ	θ	θ	θ
a	θ	θ	a	θ
b	θ	a	b	a
c	θ	θ	a	θ

Then $J = \{\theta, a\}$ ia an ideal of S, $F_J(S) = \{b\}$ and $F'_J(S) = \{\theta, c\}$. Since $bc = a$, we have that $F'_J(S)$ is not an ideal of S. Note furthermore that S is commutative, thus it is (MN) and hence pseudo-nilpotent.

Recall that if B is a band, i.e. a semigroup of idempotents, then a semigroup S is said to be a B-band union of subsemigroups S_b, with $b \in B$, provided that $S = \bigcup_{b\in B} S_b$, a disjoint union and $S_{b_1}S_{b_2} \subseteq S_{b_1b_2}$ for all $b_1, b_2 \in B$.

Lemma 3.2. *Let S be a pseudo-nilpotent finite semigroup. Assume J is an ideal of S and $J = \mathcal{M}^0(G, I, \Lambda; P)$, a regular Rees matrix semigroup. Assume J is not nilpotent. The following properties hold.*

1. *$F_J = \{s \in S\backslash J \mid sj \neq \theta, js \neq \theta, j'sj \neq \theta$ for all $j, j' \in J\backslash\{\theta\}\}$ and hence $F_J \cup (J\backslash\{\theta\})$ is a subsemigroup of S and $F'_J(S)$ is an ideal of S.*
2. *If $a \in F_J \cup (J\backslash\{\theta\})$, then there exists a unique pair $(e, f) = (e(a), f(a)) \in I \times \Lambda$ such that there is no edge in $\mathcal{N}_S$ between a and $(g; e, f)$ for every $g \in G$, i.e. $\langle (g; e, f), a\rangle$ is nilpotent for all $g \in G$ and there is an edge in $\mathcal{N}_S$ between a and each element in $J\backslash(G_{e,f} \cup \{\theta\})$. Furthermore,*

$$\Phi : S \longrightarrow (I \times \Lambda)^0$$

with

$$a \mapsto \begin{cases} (e(a), f(a)) & \text{if } a \in F_J \cup (J\backslash\{\theta\}) \\ \theta & \text{otherwise} \end{cases}$$

is a semigroup epimorphism from S to the rectangular band $(I \times \Lambda)^0$. We call Φ a pseudo-nilpotent homomorphism. Also we have $a(J\backslash\{\theta\}) \subseteq G_{e(a),*}$ and $(J\backslash\{\theta\})a \subseteq G_{*,f(a)}$ for $a \in F_J \cup (J\backslash\{\theta\})$.

(3) *S is the $\Phi(S)$-band union of the semigroups $\Phi^{-1}(\Phi(a))$ with $a \in S$, that is $S = \bigcup_{a\in S} \Phi^{-1}(\Phi(a))$ and $\Phi^{-1}(\Phi(a))\,\Phi^{-1}(\Phi(b)) \subseteq \Phi^{-1}(\Phi(ab))$. The sets $\Phi^{-1}(\Phi(a))$ will be called the pseudo-nilpotent classes of S.*

(4) *If $a, b \in F_J \cup (J\backslash\{\theta\})$ and $\Phi(a) \neq \Phi(b)$, then there is an edge in $\mathcal{N}_S$ between a and b.*

(5) *$(S\backslash F_J)/J$ is an ideal of S/J and J is an ideal of $F_J \cup J$.*

Proof. (1) Because J is pseudo-nilpotent, Lemma 3.1 yields that all entries of P are non-zero. Assume $a \in F_J \cup (J\backslash\{\theta\})$. Then there exists $(g; i, \lambda) \in J$ such that $a(g; i, \lambda) \neq \theta$ or $(g; i, \lambda)a \neq \theta$. Since all entries of P are non-zero, we get that aj, ja, jaj' are non-zero for all $j, j' \in J\backslash\{\theta\}$. Then

$$F_J = \{s \in S\backslash J \mid sj \neq \theta,\ js \neq \theta,\ j'sj \neq \theta \text{ for all } j, j' \in J\backslash\{\theta\}\}.$$

Hence $F_J \cup (J\backslash\{\theta\})$ is a subsemigroup of S.

Now let $a \in F'_J(S)$. Suppose that there exist $b \in S$ such that $ab \notin F'_J(S)$. Thus $ab \in F_J \cup (J\backslash\{\theta\})$. Since $F_J \cup (J\backslash\{\theta\})$ is a subsemigroup of S and $J\backslash\{\theta\} \neq \varnothing$, there exists $j \in J\backslash\{\theta\}$ such that $abj \neq \theta$. Therefore, $a \in F_J \cup (J\backslash\{\theta\})$ and we get a contradiction. Hence $F'_J(S)$ is a right ideal of S. Similarly we can prove that $F'_J(S)$ is a left ideal. Thus $F'_J(S)$ is an ideal of S.

(2) Let $a \in F_J \cup (J\backslash\{\theta\})$. The group G is isomorphic with a subsemigroup of S. Hence G is pseudo-nilpotent too. Hence, by Lemma 2.5, G is nilpotent. Because, by assumption J is not nilpotent and G is nilpotent, we have $|I| > 1$ or $|\Lambda| > 1$. We suppose that $|I| > 1$.

Let $(g; i, j) \in J$ with $g \in G$. Since J is an ideal of S, then there exist $i', i'' \in I$, $\lambda', \lambda'' \in \Lambda$ such that $a(g; i, \lambda) = (g'; i', \lambda), a^2(g; i, \lambda) = (g''; i'', \lambda), (g; i, \lambda)a = (k; i, \lambda')$ and $(g; i, \lambda)a^2 = (k'; i, \lambda'')$ for some $g', g'', k, k' \in G$. Therefore, for every $1 \leq n$,

$$(\lambda_{2n}(a, a^2, (g; i, \lambda), 1, \ldots, 1), \rho_{2n}(a, a^2, (g; i, \lambda), 1, \ldots, 1)) =$$
$$((m_{2n}; i', \lambda'), (m'_{2n}; i'', \lambda'')),$$
$$(\lambda_{2n-1}(a, a^2, (g; i, \lambda), 1, \ldots, 1), \rho_{2n-1}(a, a^2, (g; i, \lambda), 1, \ldots, 1)) =$$
$$((m_{2n-1}; i', \lambda''), (m'_{2n-1}; i'', \lambda')),$$

for some $m_i, m'_i \in G$. If $(i', \lambda') \neq (i'', \lambda'')$ (in particular $a \neq a^2$) then

$$\lambda_{n'}(a, a^2, (g; i, \lambda), 1, \ldots, 1) \neq \rho_{n'}(a, a^2, (g; i, \lambda), 1, \ldots, 1),$$

for every $n' \geq 1$. As S is finite, there exist distinct positive integers t and r such that

$$(\lambda_t(a, a^2, (g; i, \lambda), 1, \ldots, 1), \rho_t(a, a^2, (g; i, \lambda), 1, \ldots, 1)) =$$
$$(\lambda_r(a, a^2, (g; i, \lambda), 1, \ldots, 1), \rho_r(a, a^2, (g; i, \lambda), 1, \ldots, 1)).$$

Because S is pseudo-nilpotent, we thus obtain an edge in $\mathcal{N}_S$ between a and a^2 , a contradiction. Therefore $i' = i''$ and $\lambda' = \lambda''$.

Let $r \in G$. Then there exist $w, z, r', r'' \in G$ such that

$$a(r;i',\lambda') = a(g';i',\lambda)(r';i,\lambda') = aa(g;i,\lambda)(r';i,\lambda') =$$
$$(g'';i',\lambda)(r';i,\lambda') = (w;i',\lambda'),$$
$$(r;i',\lambda')a = (r'';i',\lambda)(k;i,\lambda')a = (r'';i',\lambda)(g;i,\lambda)aa =$$
$$(r'';i',\lambda)(k';i,\lambda') = (z;i',\lambda').$$

Hence $G_{i',\lambda'}$ is an ideal of the semigroup $\langle a\rangle \cup G_{i',\lambda'}$. Since $G \cong G_{i',\lambda'}$ is nilpotent, Lemma 2.7 yields that there is no edge in $\mathcal{N}_S$ between a and any element of $G_{i',\lambda'}$.

Now let $(s;\alpha,\beta) \in J, (\alpha,\beta) \neq (i',\lambda')$. We have, for every $1 \le n$,

$$(\lambda_{2n}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1), \rho_{2n}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1)) =$$
$$((v_{2n};i',\lambda'),(v'_{2n};\alpha,\beta)),$$
$$(\lambda_{2n-1}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1), \rho_{2n-1}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1)) =$$
$$((v_{2n-1};i',\beta),(v'_{2n-1};\alpha,\lambda')),$$

for some $v_i, v'_i \in G$. Because $(\alpha,\beta) \neq (i',\lambda')$, we thus have that

$$\lambda_{n'}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1) \neq \rho_{n'}(a,(s;\alpha,\beta),(1;i',\lambda'),1,\ldots,1),$$

for every $n' \ge 1$. In a similar way as above for a and a^2, there is an edge in $\mathcal{N}_S$ between $(s;\alpha,\beta)$ and a for every $s \in G$. Therefore, between a and all elements of $J\backslash(G_{i',\lambda'} \cup \{\theta\})$ there are edges in $\mathcal{N}_S$ and (i',λ') is the unique element of $I \times \Lambda$ such that there is no edge in $\mathcal{N}_S$ between a and all elements of $G_{i',\lambda'}$.

So we have shown that if $a \in F_J$ and $a(g;i,\lambda) = (g';i',\lambda), (g;i,\lambda)a = (k';i,\lambda')$ then $\langle a,(g;i',\lambda')\rangle$ is nilpotent for any $g \in G$ and $\langle a,(g;\alpha,\beta)\rangle$ is not nilpotent for any $g \in G$ and $(\alpha,\beta) \neq (i',\lambda')$. Since $\theta \neq a(J\backslash\{\theta\})$ and $\theta \neq (J\backslash\{\theta\})a$, it follows that $a(J\backslash\{\theta\}) \subseteq G_{i',*}$ and $(J\backslash\{\theta\})a \subseteq G_{*,\lambda'}$. This fact will be used twice in the proof.

Note that if $(i,\lambda) \neq (i',\lambda')$ then it is readily verified that $\langle(g;i,\lambda),(g';i',\lambda')\rangle$ is not nilpotent. So there is an edge in $\mathcal{N}_S$ between $(g;i,\lambda)$ and $(g';i',\lambda')$. On the other hand, each $\langle(g;i,\lambda),(g';i,\lambda)\rangle$ is nilpotent (as a subsemigroup of $G_{i,\lambda} \cong G$).

Because of the above, we now can define a function

$$\Phi : S \to (I \times \Lambda)^0$$

as follows. For $a \in S\backslash F'_J, \Phi(a) = (i',\lambda')$ where $(i',\lambda') \in I \times \Lambda$ is such that $a(J\backslash\{\theta\}) \subseteq G_{i',*}$ and $(J\backslash\{\theta\})a \subseteq G_{*,\lambda'}$. If $a \in F'_J$ then we define $\Phi(a) = \theta$.

Consider $I \times \Lambda$ as a rectangular band (for the natural multiplication). Hence $(I \times \Lambda)^0$ is a band with a zero θ.

We claim that Φ is a semigroup homomorphism.

So let $x, y \in S$. If $x \in F'_J$ or $y \in F'_J$, then $xy \in F'_J$ (as by part (1), F'_J is an ideal of S); hence $\Phi(xy) = \Phi(x)\Phi(y) = \theta$.

Assume now that $x, y \in S\backslash F'_J$. Then there exist unique $(i_1, \lambda_1), (i_2, \lambda_2) \in I \times \Lambda$, such that $x(J\backslash\{\theta\}) \subseteq G_{i_1,*}, (J\backslash\{\theta\})x \subseteq G_{*,\lambda_1}, y(J\backslash\{\theta\}) \subseteq G_{i_2,*}$ and $(J\backslash\{\theta\})y \subseteq G_{*,\lambda_2}$. Consequently, $xy(J\backslash\{\theta\}) \subseteq xG_{i_2,*} \subseteq G_{i_1,*}$ and $(J\backslash\{\theta\})xy \subseteq G_{*,\lambda_1}y \subseteq G_{*,\lambda_2}$. Hence, by the fact mentioned above, $\Phi(xy) = (i_1, \lambda_2) = (i_1, \lambda_1)(i_2, \lambda_2) = \Phi(x)\Phi(y)$. So, indeed, Φ is a semigroup homomorphism.

(3) Because each element of $(I \times \Lambda)^0$ is idempotent, one has that $\Phi^{-1}(\Phi(a))$ is a subsemigroup of S for each $a \in S$. As Φ is surjective, we get that $S = \bigcup_{x \in \Phi(S)} \Phi^{-1}(x)$, a disjoint union and $\Phi^{-1}(x)\Phi^{-1}(y) \subseteq \Phi^{-1}(xy)$. Hence statement (3) follows.

(4) Assume $a, b \in F_J \cup (J\backslash\{\theta\})$ are such that $(i, \lambda) = \Phi(a) \neq \Phi(b) = (i', \lambda')$. We have, for every $1 \leq n$,

$$(\lambda_{2n}(a, b, (1; i', \lambda'), 1, \ldots, 1), \rho_{2n}(a, b, (1; i', \lambda'), 1, \ldots, 1))$$
$$= ((m_{2n}; i, \lambda), (m'_{2n}; i', \lambda'),$$
$$(\lambda_{2n-1}(a, b, (1; i', \lambda'), 1, \ldots, 1), \rho_{2n-1}(a, b, (1; i', \lambda'), 1, \ldots, 1)) =$$
$$((m_{2n-1}; i, \lambda'), (m'_{2n-1}; i', \lambda)),$$

for some $m_k, m'_k \in G$. Because $(i, \lambda) \neq (i', \lambda')$,

$$\lambda_{n'}(a, b, (1; i', \lambda'), 1, \ldots, 1) \neq \rho_{n'}(a, b, (1; i', \lambda'), 1, \ldots, 1),$$

for every $n' \geq 1$. Since S is finite, there exist distinct positive integers t and r such that

$$(\lambda_t(a, b, (1; i', \lambda'), 1, \ldots, 1), \rho_t(a, b, (1; i', \lambda'), 1, \ldots, 1)) =$$
$$(\lambda_r(a, b, (1; i', \lambda'), 1, \ldots, 1), \rho_r(a, b, (1; i', \lambda'), 1, \ldots, 1)).$$

As S is pseudo-nilpotent, we obtain that there is an edge in $\mathcal{N}_S$ between a and b.

(5) By part (1), F'_J is an ideal of S. Now since J is an ideal of S, statement (5) is obvious. □

Every finite semigroup S has principal series:

$$S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \varnothing.$$

That is, each S_i is an ideal of S and there is no ideal of S strictly between S_i and S_{i+1} (for convenience we call the empty set an ideal of S). Each principal factor $S_i/S_{i+1} (1 \leq i \leq m)$ of S either is completely 0-simple, completely simple or null. Every completely 0-simple factor semigroup is isomorphic with a regular Rees matrix semigroup over a finite group G.

Suppose S is pseudo-nilpotent. Then, by Lemma 3.1, every principal factor which is a regular Rees matrix semigroup is nilpotent or all entries of the respective sandwich matrix are non-zero, that is, it is a union of groups. If S_m and all S_i/S_{i+1} are nilpotent, then by Lemma 2.3, S is nilpotent as well.

Definition 3.3. *If S_i/S_{i+1} is not nilpotent (thus $S_i\backslash S_{i+1}$ is a semigroup) and there is no edge in $\mathcal{N}_S$ between any element of $S_i\backslash S_{i+1}$ and any element of S_{i+1} then we say that $S_i\backslash S_{i+1}$ is a root of S.*

In case $S_i\backslash S_{i+1}$ is a root of S, then it follows from Lemma 3.2.(2) that if $i' > i$ and $S_i\backslash S_{i+1} \subseteq F_{S_{i'}/S_{i'+1}}(S/S_{i'+1})$ then $S_{i'}/S_{i'+1}$ is nilpotent.

Note that there exist pseudo-nilpotent finite semigroups with more than one root. An example is the semigroup $T = \{a, b, c, d, e\}$ with multiplication table

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	d	a
d	a	a	c	d	a
e	a	e	a	a	e

The subsemigroups $\{b, e\}$ and $\{c, d\}$ are roots of T.

It is now convenient to identify the non-zero elements of S/I with those of $S\backslash I$ for I an ideal of S. Let $j < i \leq m$ and let $B = F'_{S_i/S_{i+1}}(S/S_{i+1}) \cap (S_j\backslash S_{j+1})$. We claim that $B \cup S_{j+1}$ is an ideal of S.

Indeed, let $x \in B \cup S_{j+1}$ and let $a \in S$. We need to show that $ax \in B \cup S_{j+1}$ and $xa \in B \cup S_{j+1}$. We only prove the former, the other one is shown similar. We give a proof by contradiction. So suppose $ax \notin B \cup S_{j+1}$. Since $x \in S_j$ and S_j is an ideal of S, we have $ax \in S_j\backslash S_{j+1}$. Furthermore, $ax \notin B$ and $ax \notin S_i\backslash S_{i+1}$ imply $ax \in F_{S_i/S_{i+1}}(S/S_{i+1})$ and thus there exists an element y in $S_i\backslash S_{i+1}$ such that $axy \notin S_{i+1}$ or $yax \notin S_{i+1}$. Clearly $ya \in S_i$. Now since x in B, $x \in F'_{S_i/S_{i+1}}(S/S_{i+1})$ and thus xy and yax are in S_{i+1}. However, this is in contradiction with $axy \notin S_{i+1}$ or $yax \notin S_{i+1}$. This proves that indeed $ax \in B \cup S_{j+1}$.

Now, since $S_{j+1} \subseteq B \cup S_{j+1} \subseteq S_j$ and because there is no ideal strictly between S_{j+1} and S_j, we get $B = \varnothing$ or $B = S_j\backslash S_{j+1}$. It therefore follows that if $B \neq \varnothing$ then $S_j\backslash S_{j+1} \subseteq F'_{S_i/S_{i+1}}(S/S_{i+1})$. On the other hand if $B = \varnothing$ then $S_j\backslash S_{j+1} \subseteq F_{S_i/S_{i+1}}(S/S_{i+1})$.

For $1 \leq i \leq m$ let

$$i^* = \{j \leq i \mid S_j\backslash S_{j+1} \subseteq F_{S_i/S_{i+1}}(S/S_{i+1}) \cup S_i\backslash S_{i+1}\}$$

and

$$S^{(i)} = \bigcup_{j \in i^*} (S_j\backslash S_{j+1}).$$

Since $F_{S_i/S_{i+1}}(S/S_{i+1}) \subseteq S\backslash S_i$, $S\backslash S_i = \bigcup_{1\leq k<i}(S_k\backslash S_{k+1})$ and the fact mentioned above we have

$$S^{(i)} = F_{S_i/S_{i+1}}(S/S_{i+1}) \cup S_i\backslash S_{i+1}.$$

Definition 3.4. *If $S_i\backslash S_{i+1}$ is a root then the set $S^{(i)}$ is called the stem of $S_i\backslash S_{i+1}$. In this case, because of Lemma 3.2, if $S_j\backslash S_{j+1} \subseteq S^{(i)}$ then there is a path in $\mathcal{N}_S$ between any two elements $s, t \in S^{(i)}$.*

Note that for every set $T = S_i\backslash S_{i+1}$ we have three possible cases: (i) T is a root, (ii) T is not a root and there exists a non-nilpotent semigroup $S_j\backslash S_{j+1}$ such that $T \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$ (so in this case there is an edge in

$\mathcal{N}_S$ between some element of T and some element of $S_j \backslash S_{j+1}$), (iii) T is not a root and if $T \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$ then S_j/S_{j+1} is nilpotent. The third case will be given a special name.

Definition 3.5. *We say that $S_i \backslash S_{i+1}$ is an isolated subset provided that $S_i \backslash S_{i+1}$ is not a root that satisfies the property that if $S_i \backslash S_{i+1} \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$ then S_j/S_{j+1} is nilpotent.*

Definition 3.6. *Suppose $S^{(i_1)}$ and $S^{(i_2)}$ are two distinct stems of S. If $S^{(i_1)} \cap S^{(i_2)} \neq \varnothing$ then $S_j \backslash S_{j+1} \subseteq S^{(i_1)} \cap S^{(i_2)}$ for some j. We call $S_j \backslash S_{j+1}$ a connection between the stems $S^{(i_1)}$ and $S^{(i_2)}$.*

The reason for this name is clear as in the upper non-nilpotent graph $\mathcal{N}_S$ there is a path from any element of $S^{(i_1)}$ to any element of $S^{(i_2)}$ via a vertex in $S_j \backslash S_{j+1}$. For example, in the semigroup $T = \{\theta, a, b, c, d, f\}$ with multiplication table

	θ	a	b	c	d	f
θ	θ	θ	θ	θ	θ	θ
a	θ	a	b	θ	θ	b
b	θ	a	b	θ	θ	b
c	θ	θ	θ	c	d	d
d	θ	θ	θ	c	d	d
f	θ	a	b	c	d	f

We prove that T is pseudo-nilpotent by contradiction. So, suppose the contrary. Hence, there exist distinct elements $x, y \in T$, elements $w_1, \ldots, w_m \in T^1$, an ideal I of $\langle x, y, w_1, \ldots, w_m \rangle$ such that

$$\lambda_t(x, y, w_1, w_2, \ldots, w_t) = \lambda_m(x, y, w_1, w_2, \ldots, w_m),$$

$$\rho_t(x, y, w_1, w_2, \ldots, w_t) = \rho_m(x, y, w_1, w_2, \ldots, w_m),$$

$\lambda_m \neq \rho_m$, $\lambda_m, \rho_m \notin I$ and $\langle \lambda_i, \rho_i \rangle$ is nilpotent in $\langle x, y, w_1, \ldots, w_m \rangle / I$ for some $0 \leq i \leq m$. Because the subsemigroups $\{\theta, a, b\}$ and $\{\theta, c, d\}$ are ideals of S and $\{\theta, a, b\}\{\theta, c, d\} = \{\theta, c, d\}\{\theta, a, b\} = \{\theta\}$, the equalities imply that neither $\lambda_i \in \{\theta, a, b\}$, $\rho_i \in \{\theta, c, d\}$ nor $\lambda_i \in \{\theta, c, d\}$, $\rho_i \in \{\theta, a, b\}$. Since $\langle \lambda_i, \rho_i \rangle$ is nilpotent in $\langle x, y, w_1, \ldots, w_m \rangle / I$, either $\{\lambda_i, \rho_i\} = \{b, f\}$ or $\{\lambda_i, \rho_i\} = \{d, f\}$. Suppose that $\{\lambda_i, \rho_i\} = \{b, f\}$. Since $\theta \notin \{\lambda_{i+1}, \rho_{i+1}\}$, $w_{i+1} \in \{a, b, f, 1\}$. Then $\lambda_{i+1} = \rho_{i+1} = b$, a contradiction. Similarly one obtains a contradiction for $\{\lambda_i, \rho_i\} = \{d, f\}$. So, indeed, T is pseudo-nilpotent.

Note that the set $\{f\}$ is a connection between the stems $\{a, b, f\}$ and $\{c, d, f\}$.

Let S be a pseudo-nilpotent finite semigroup with principal series

$$S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \varnothing.$$

If S_i/S_{i+1} is not nilpotent then $(S_i/S_{i+1})^0 = \mathcal{M}^0(G, I, \Lambda; P)$ (with all entries of P non-zero) and we denote by $\Phi_i : (S/S_{i+1})^0 \to (I \times \Lambda)^0$ the pseudo-nilpotent homomorphism obtained in Lemma 3.2.

Theorem 3.7. *Let S be a pseudo-nilpotent finite semigroup with principal series*

$$S = S_1 \supset S_2 \supset \ldots \supset S_m \supset S_{m+1} = \varnothing.$$

The following properties hold.

(1) *The union K of all isolated subsets $S_i \backslash S_{i+1}$ is the largest nilpotent ideal of S and it is the set of all isolated vertices in $\mathcal{N}_S$ (possibly $K = \varnothing$).*
(2) *If $S_{i_1} \backslash S_{i_1+1} \subseteq S^{(i_2)}$ and $S_{i_2} \backslash S_{i_2+1} \subseteq S^{(i_3)}$, then $S_{i_1} \backslash S_{i_1+1} \subseteq S^{(i_3)}$. If $S_{i_2} \backslash S_{i_2+1} = \mathcal{M}(G_2, I_2, \Lambda_2; P_2)$ and $S_{i_3} \backslash S_{i_3+1} = \mathcal{M}(G_3, I_3, \Lambda_3; P_3)$ are not nilpotent and $\Phi_2 : (S/S_{i_2+1})^0 \longrightarrow (I_2 \times \Lambda_2)^0$, $\Phi_3 : (S/S_{i_3+1})^0 \longrightarrow (I_3 \times \Lambda_3)^0$ are pseudo-nilpotent homomorphisms, then $\Phi_3(a) = \Phi_3((g; \Phi_2(a)))$, for every $a \in S_{i_1} \backslash S_{i_1+1}$ and $g \in G_2$.*
(3) *If $S_i \backslash S_{i+1}$ is not an isolated subset and not a root, then there exists a root $S_j \backslash S_{j+1}$ such that $i < j$ and $S_i \backslash S_{i+1} \subseteq S^{(j)}$.*
(4) *$S \backslash K = \bigcup S^{(i)}$, where the union runs over all i with $S_i \backslash S_{i+1}$ a root.*
(5) *Every stem $S^{(i)}$ is a subsemigroup.*
(6) *$(S_i \backslash S_{i+1})(S_j \backslash S_{j+1}) \subseteq K$ if and only if $S_i \backslash S_{i+1}$ and $S_j \backslash S_{j+1}$ are not in a same stem.*

Proof. (1) First suppose that S does not have any isolated subset. Then for every principal factor S_i/S_{i+1}, the subset $S_i \backslash S_{i+1}$ is a root or there exists non-nilpotent semigroup $S_j \backslash S_{j+1}$ such that $S_i \backslash S_{i+1} \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$. Because of Lemma 3.2.(2), in both cases, all elements of $S_i \backslash S_{i+1}$ are non-isolated vertices in $\mathcal{N}_S$. Therefore $\mathcal{N}_S$ has no isolated vertex. Now suppose that I is a nilpotent ideal of S. Then $\mathcal{N}_I$ is empty and by Lemma 2.7 the elements of I are isolated vertices in $\mathcal{N}_S$. Hence, by the above, I is empty.

Now assume S has an isolated subset, i.e. we suppose that $K \neq \varnothing$. Suppose that $a \in K, b \in S$ and $ab \notin K$. Let i and j be such that $a \in S_i \backslash S_{i+1}$ and $ab \in S_j \backslash S_{j+1}$. Because each S_k is an ideal of S, it is clear that $i < j$. Since $ab \notin K$ we have that $S_j \backslash S_{j+1}$ is not an isolated subset. So either (i) $S_j \backslash S_{j+1}$ is a root, or (ii) $S_j \backslash S_{j+1}$ is not a root and $S_j \backslash S_{j+1} \subseteq F_{S_k/S_{k+1}}(S/S_{k+1})$ for some k with $j < k$ and S_k/S_{k+1} is not nilpotent.

We first show that case (i) leads to a contradiction. So assume $S_j \backslash S_{j+1}$ is a root. In particular, S_j/S_{j+1} is not nilpotent and $|S_j \backslash S_{j+1}| > 1$. Thus there exists $x \in S_j \backslash S_{j+1}$ such that $x \neq ab$. Because $a \in S_i \backslash S_{i+1}$ and since $S_i \backslash S_{i+1}$ is an isolated subset we get that $a \in F'_{S_j/S_{j+1}}(S/S_{j+1})$. Hence $xa \in S_{j+1}$ and thus also $xab \in S_{j+1}$. On the other hand, by Lemma 3.1, $S_j \backslash S_{j+1}$ is a semigroup. But then $x, ab \in S_j \backslash S_{j+1}$ implies that $xab \in S_j \backslash S_{j+1}$. This contradicts with $xab \in S_{j+1}$ and $S_j \backslash S_{j+1}$ being a root.

Next we show that case (ii) also leads to a contradiction. So suppose that $S_j \backslash S_{j+1}$ is not a root and that there exists a positive integer k such that $ab \in F_{S_k/S_{k+1}}(S/S_{k+1})$, $j < k$ and S_k/S_{k+1} is not nilpotent. Choose $y \in S_k \backslash S_{k+1}$. Since $S_i \backslash S_{i+1}$ is an isolated subset and because $a \in S_i \backslash S_{i+1}$ we get that $a \in F'_{S_k/S_{k+1}}(S/S_{k+1})$. Hence, $ya \in S_{k+1}$. As S_{k+1} is an ideal of S, we

thus obtain that $yab \in S_{k+1}$. So, by Lemma 3.2.(1), $ab \notin F_{S_k/S_{k+1}}(S/S_{k+1})$, a contradiction.

We thus have shown that indeed K is a right ideal of S. Similarly one shows that it is a left ideal and thus it is an ideal.

We now prove that all elements of K are isolated vertices. So suppose the contrary and let i be the largest positive integer such that $S_i \backslash S_{i+1} \subseteq K$ and $S_i \backslash S_{i+1}$ contains a non-isolated vertex, say v. Then there exists an element $w \in S$ such that $\langle v, w \rangle$ is not nilpotent. Lemma 2.1 implies that there exists a positive integer m', distinct elements $x, y \in \langle v, w \rangle$ and elements $w_1, w_2, \ldots, w_{m'} \in \langle v, w \rangle^1$, such that $x = \lambda_{m'}(x, y, w_1, w_2, \ldots, w_{m'})$, $y = \rho_{m'}(x, y, w_1, w_2, \ldots, w_{m'})$. As S is pseudo-nilpotent, we get that $\langle x, y \rangle$ is not nilpotent. Hence, since $\langle w \rangle$ is nilpotent, we get that $\{x, y\} \nsubseteq \langle w \rangle$. As S_i is an ideal of S and $v \in S_i$, we clearly have $\langle w, v \rangle \backslash \langle w \rangle \subseteq S_i$. Therefore $\{x, y\} \cap S_i \neq \varnothing$. Again because S_i is an ideal of S and since $m' \geq 1$, the equalities

$$x = \lambda_{m'}(x, y, w_1, w_2, \ldots, w_{m'}),\ y = \rho_{m'}(x, y, w_1, w_2, \ldots, w_{m'}),$$

imply that $x, y \in S_i$. Because K is an ideal we obtain in a similar manner that $x, y \subseteq K$. By the maximality choice of i we have that $x, y \in S_i \backslash S_{i+1}$. Since S is pseudo-nilpotent, the above equalities yield that there is an edge between x and y in $\mathcal{N}_{S/S_{i+1}}$. So, by Lemma 3.1, $S_i \backslash S_{i+1}$ is a non-nilpotent semigroup.

Since $S_i \backslash S_{i+1}$ is an isolated subset and $S_i \backslash S_{i+1}$ is not nilpotent, it follows from the definition of root, that there exist $i' > i, a \in S_i \backslash S_{i+1}$ and $b \in S_{i'} \backslash S_{i'+1}$ such that there is an edge in $\mathcal{N}_S$ between a and b. Again with a similar proof as above, there exist elements a', b' in $\langle a, b \rangle \cap S_{i'}$ such that $\langle a', b' \rangle$ is not nilpotent and $a', b' \in K$. Let $i'' > i$ be such that $a' \in S_{i''} \backslash S_{i''+1}$. As K is a union of isolated subsets, it follows that $S_{i''} \backslash S_{i''+1} \subseteq K$ and $S_{i''} \backslash S_{i''+1}$ contains a non-isolated vertex. This contradicts with the maximality of i. Hence we have shown that indeed all elements of K are isolated vertices.

We now show that if $S_i \backslash S_{i+1}$ is not an isolated subset then it does not contain any isolated vertex; and hence K is indeed the set of all isolated vertices. So suppose $S_i \backslash S_{i+1}$ is not an isolated subset. Then either it is a root or $S_i \backslash S_{i+1} \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$ for some $j > i$ with S_j/S_{j+1} not nilpotent. In the former case, Lemma 3.1 yields that the upper non-nilpotent graph of $S_i \backslash S_{i+1}$ is non-empty, connected and regular. Hence $S_i \backslash S_{i+1}$ does not have any isolated vertices in $\mathcal{N}_S$. In the second case, again by Lemma 3.1, there exists a $j > i$ such that $S_j \backslash S_{j+1} = \mathcal{M}(G, I, \Lambda; P)$, G a nilpotent group, $|I| + |\Lambda| \geq 3$, all entries of P are non-zero and $S_i \backslash S_{i+1} \subseteq F_{S_j/S_{j+1}}(S/S_{j+1})$. Again $\mathcal{N}_{S_j \backslash S_{j+1}}$ is a non-empty connected and regular graph. By Lemma 3.2, there exists a pseudo-nilpotent homomorphism Φ from $(S/S_{j+1})^0$ to the rectangular band $(I \times \Lambda)^0$. Furthermore, there is an edge in $\mathcal{N}_S$ between $a \in S_i \backslash S_{i+1}$ and any element in $(S_j \backslash S_{j+1}) \backslash G_{\Phi(a)}$. Hence, $S_i \backslash S_{i+1}$ does not have any isolated vertices. So, indeed, K is the set of all isolated vertices.

As $\mathcal{N}_K$ is empty and K is pseudo-nilpotent, by Lemma 2.4, the semigroup K is nilpotent. It remains to show that K is the largest nilpotent ideal of S. To do so, let $a \in K' \backslash K$ with K' an ideal that is nilpotent. Then there exists $b \in S$ with $\langle a, b\rangle$ not nilpotent. Since S is pseudo-nilpotent and $\langle a, b\rangle \subseteq \langle b\rangle \cup K'$, it follows with an argument as above that there exist $e, f \in K'$ with $\langle e, f\rangle$ not nilpotent. However, this contradicts with K' being nilpotent. So, indeed K is the largest nilpotent ideal of S.

(2) If $i_1 = i_2$ or $i_2 = i_3$, then the statement is obvious. Assume $i_1 \neq i_2$ and $i_2 \neq i_3$. Then the sets $F_{S_{i_2}/S_{i_2+1}}(S/S_{i_2+1}) \cap (S_{i_1} \backslash S_{i_1+1})$ and $F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$ $\cap (S_{i_2} \backslash S_{i_2+1})$ are not empty. Hence we get that $S_{i_1} \backslash S_{i_1+1} \subseteq F_{S_{i_2}/S_{i_2+1}}(S/S_{i_2+1})$ and $S_{i_2} \backslash S_{i_2+1} \subseteq F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$. It follows that for $a \in S_{i_1} \backslash S_{i_1+1}$ there exists $b \in S_{i_2} \backslash S_{i_2+1}$ such that $ab \in S_{i_2} \backslash S_{i_2+1}$ or $ba \in S_{i_2} \backslash S_{i_2+1}$. Suppose that $ab \in S_{i_2} \backslash S_{i_2+1}$. Then $ab \in F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$ and thus there exist $c \in S_{i_3} \backslash S_{i_3+1}$ such that $abc \in S_{i_3} \backslash S_{i_3+1}$ or $cab \in S_{i_3} \backslash S_{i_3+1}$. If $abc \in S_{i_3} \backslash S_{i_3+1}$, then $bc \in S_{i_3} \backslash S_{i_3+1}$ and thus $a \in F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$. Also if $cab \in S_{i_3} \backslash S_{i_3+1}$, then $ca \in S_{i_3} \backslash S_{i_3+1}$ and thus $a \in F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$. Similarly if $ba \in S_{i_2} \backslash S_{i_2+1}$, then $a \in F_{S_{i_3}/S_{i_3+1}}(S/S_{i_3+1})$. Consequently $S_{i_1} \backslash S_{i_1+1} \subseteq S^{(i_3)}$.

Now assume $S_{i_2} \backslash S_{i_2+1} = \mathcal{M}(G_2, I_2, \Lambda_2; P_2)$ and $S_{i_3} \backslash S_{i_3+1} = \mathcal{M}(G_3, I_3, \Lambda_3; P_3)$ are semigroups that are not nilpotent. By Lemma 3.1, $|I_2| + |\Lambda_2| > 2, |I_3| + |\Lambda_3| > 2$, G_2 and G_3 are nilpotent groups and all entries of both P_2 and P_3 are non-zero. Because of Lemma 3.2.(2) there exist the pseudo-nilpotent homomorphisms $\Phi_2 : (S/S_{i_2+1})^0 \to (I_2 \times \Lambda_2)^0$ and $\Phi_3 : (S/S_{i_3+1})^0 \to (I_3 \times \Lambda_3)^0$.

For every $a \in S_{i_1} \backslash S_{i_1+1}$ and $g \in G_2, k, l \in G_3$, by Lemma 3.2.(2), there exist elements $k', k'', l', l'' \in G_3, g', g'' \in G_2$ such that

$$\begin{aligned}
(k; \Phi_3((g; \Phi_2(a)))) &= (g; \Phi_2(a))(k'; \Phi_3((g; \Phi_2(a)))),\\
a(g; \Phi_2(a)) &= (g'; \Phi_2(a)),\\
(g'; \Phi_2(a))(k'; \Phi_3((g'; \Phi_2(a)))) &= (k''; \Phi_3((g'; \Phi_2(a)))),\\
(l; \Phi_3((g; \Phi_2(a)))) &= (l'; \Phi_3((g; \Phi_2(a))))(g; \Phi_2(a)),\\
(g; \Phi_2(a))a &= (g''; \Phi_2(a)),\\
(l'; \Phi_3((g''; \Phi_2(a))))(g''; \Phi_2(a)) &= (l''; \Phi_3((g''; \Phi_2(a)))).
\end{aligned}$$

As G_2 is nilpotent, there is no edge in $\mathcal{N}_S$ between $(g; \Phi_2(a))$, $(g'; \Phi_2(a))$ and $(g''; \Phi_2(a))$. Lemma 3.2.(4) implies that $\Phi_3((g; \Phi_2(a))) = \Phi_3((g'; \Phi_2(a))) = \Phi_3((g''; \Phi_2(a)))$. Therefore we have

$$\begin{aligned}
a(k; \Phi_3((g; \Phi_2(a)))) &= a(g; \Phi_2(a))(k'; \Phi_3((g; \Phi_2(a))))\\
&= (g'; \Phi_2(a))(k'; \Phi_3((g; \Phi_2(a))))\\
&= (k''; \Phi_3((g; \Phi_2(a)))),\\
(l; \Phi_3((g; \Phi_2(a))))a &= (l'; \Phi_3((g; \Phi_2(a))))(g; \Phi_2(a))a\\
&= (l'; \Phi_3((g; \Phi_2(a))))(g''; \Phi_2(a))\\
&= (l''; \Phi_3((g; \Phi_2(a)))).
\end{aligned}$$

It implies that $\Phi_3(a) = \Phi_3((g;\Phi_2(a)))$.

(3) Since S is finite and because $S_i\backslash S_{i+1}$ is not an isolated subset and not a root, there exists a positive integer i' such that $S_i\backslash S_{i+1} \subseteq F_{S_{i'}/S_{i'+1}}(S/S_{i'+1})$, $S_{i'}/S_{i'+1}$ is not nilpotent and if $S_i\backslash S_{i+1} \subseteq F_{S_{i''}/S_{i''+1}}(S/S_{i''+1})$ for some $i'' > i'$, then $S_{i''}/S_{i''+1}$ is nilpotent. Clearly $S_i\backslash S_{i+1} \subseteq S^{(i')}$.

If $S_{i'}\backslash S_{i'+1}$ is a root, then the statement obviously holds. Otherwise, as $S_{i'}\backslash S_{i'+1}$ is not nilpotent (and thus its vertices in $\mathcal{N}_S$ are not isolated), we obtain from part (1) that $S_{i'}\backslash S_{i'+1}$ is not an isolated subset. Hence there exists a positive integer $j > i'$ such that $S_{i'}\backslash S_{i'+1} \subseteq S^{(j)}$ and $S_j\backslash S_{j+1}$ is not nilpotent. By part (2), $S_i\backslash S_{i+1} \subseteq S^{(j)}$. However, this contradicts with the condition on i'.

(4) Let $s \in S\backslash K$ and let i be such that $s \in S_i\backslash S_{i+1}$. In particular $S_i\backslash S_{i+1}$ is not an isolated subset. If $S_i\backslash S_{i+1}$ is a root then $s \in S^{(i)}$. If $S_i\backslash S_{i+1}$ is not a root, then, by part (3), $s \in S^{(k)}$ for some $k > i$ and $S_k\backslash S_{k+1}$ is a root. It easily can be verified that any element of a non-isolated subset is not an isolated vertex in $\mathcal{N}_S$. Hence the statement follows.

(5) Suppose $S^{(i)}$ is a stem. As $S_i\backslash S_{i+1}$ is a root, it is not nilpotent and by Lemma 3.1, $S_i\backslash S_{i+1} = \mathcal{M}(G, I, \Lambda; P)$ a regular Rees matrix semigroup with $|I| + |\Lambda| > 2$, G a nilpotent group and all entries of P non-zero and there exists a pseudo-nilpotent homomorphism $\Phi : (S/S_{i+1})^0 \to (I \times \Lambda)^0$. Now suppose that $a, b \in S^{(i)}$. Hence $\Phi(a)$ and $\Phi(b)$ are non-zero in $(I \times \Lambda)^0$ and thus $\Phi(ab) = \Phi(a)\,\Phi(b)$ is also non-zero. So $ab \in S^{(i)}$.

(6) Suppose $a \in S_i\backslash S_{i+1}, b \in S_j\backslash S_{j+1}$, $ab \notin K$ and $S_i\backslash S_{i+1}$ and $S_j\backslash S_{j+1}$ are not in a same stem. By part (4), there exists some k such that $ab \in S^{(k)}$ and $S_k\backslash S_{k+1}$ is a root. Hence $ab \notin F'_{S_k/S_{k+1}}(S/S_{k+1})$. As by Lemma 3.2.(1) $F'_{S_k/S_{k+1}}(S/S_{k+1})$ is an ideal of S/S_{k+1}, it follows that $a, b \notin F'_{S_k/S_{k+1}}(S/S_{k+1})$. Hence, $a, b \in F_{S_k/S_{k+1}}(S/S_{k+1})$. But this contradicts with the assumption that a and b do not belong to a same stem. This proves one implication of (6). The converse easily can be verified. $\square$

We now give several consequences of Theorem 3.7. First we extend Lemma 2.5 as follows.

Corollary 3.8. *A finite monoid S is pseudo-nilpotent if and only if it is nilpotent.*

Proof. Suppose S is a pseudo-nilpotent finite monoid. From Theorem 3.7 we know that the set consisting of the isolated vertices is the largest nilpotent ideal K of S. Clearly $1 \in K$. Hence, $S = K$ is nilpotent. The result follows. $\square$

Note that in general $(S_i\backslash S_{i+1})(S_j\backslash S_{j+1}) \nsubseteq K$ does not imply that if $S_i\backslash S_{i+1}$ is contained in a stem $S^{(h)}$ then $S_j\backslash S_{j+1} \subseteq S^{(h)}$. However, we can prove the following.

Corollary 3.9. *Let S be a pseudo-nilpotent finite semigroup and $a, b \in S$. If there exists an edge in $\mathcal{N}_S$ between a and b, then there exists a stem $S^{(i)}$ such that $a, b \in S^{(i)}$.*

Proof. Since a and b are not isolated vertices, by Theorem 3.7.(1), both a and b do not belong to K. Let S_i/S_{i+1} and S_j/S_{j+1} be principal factors of S such that $a \in S_i \backslash S_{i+1}$ and $b \in S_j \backslash S_{j+1}$. If ab or ba is not in K, then $(S_i \backslash S_{i+1})(S_j \backslash S_{j+1}) \nsubseteq K$ or $(S_j \backslash S_{j+1})(S_i \backslash S_{i+1}) \nsubseteq K$ and thus by Theorem 3.7.(6) the statement obviously holds.

Now suppose that $ab, ba \in K$. Since $\langle a, b \rangle$ is not nilpotent, by Lemma 2.1, there exists a positive integer m, distinct elements $x, y \in \langle a, b \rangle$ and $w_1, w_2, \ldots, w_m \in \langle a, b \rangle^1$ such that $x = \lambda_m(x, y, w_1, w_2, \ldots, w_m)$, $y = \rho_m(x, y, w_1, w_2, \ldots, w_m)$. As S is pseudo-nilpotent and $\langle a \rangle$ and $\langle b \rangle$ are nilpotent, we get that $\{x, y\} \nsubseteq \langle a \rangle$ and $\{x, y\} \nsubseteq \langle b \rangle$. Since $ab, ba \in K$ and K is an ideal, it follows that $x, y \in K$. As K is nilpotent, Theorem 3.7.(1) then implies that $\langle x, y \rangle$ is nilpotent, a contradiction. □

Corollary 3.10. *Let S be a pseudo-nilpotent finite semigroup. The following properties hold.*

1. *Every stem is connected and any two distinct elements of a stem are connected by a path of length at most 2.*
2. *If K is empty, then $S = S^{(i)}$ for some root $S_i \backslash S_{i+1}$, $\mathcal{N}_S$ is connected and every two distinct vertices are connected by a path of length at most 2.*
3. *If S does not have any connections, then every connected component with more than one element is a stem and it is a subsemigroup.*
4. *The union of two stems that have a connection is a connected subset of $\mathcal{N}_S$. Furthermore, every shortest path in this union has length at most 4.*

Proof. We use the same notation as in Theorem 3.7.

(1) Assume $S_i \backslash S_{i+1}$ is a root and suppose $s, t \in S^{(i)}$, with $s \neq t$. By Lemma 3.1, $S_i \backslash S_{i+1} = \mathcal{M}(G, I, \Lambda; P)$, a regular Rees matrix semigroup, with $|I| + |\Lambda| > 2$, G is a nilpotent group and all entries of P are non-zero. By Lemma 3.2.(2), there exists a pseudo-nilpotent homomorphism

$$\Phi : (S/S_{i+1})^0 \to (I \times \Lambda)^0.$$

Since $s, t \in S^{(i)}$, we have $\Phi(s) \neq \theta, \Phi(t) \neq \theta$.

If $\Phi(s) \neq \Phi(t)$, then by Lemma 3.2.(4), there is an edge in $\mathcal{N}_S$ between s and t. If $\Phi(s) = \Phi(t)$, there exists $(c, d) \in I \times \Lambda$ such that $(c, d) \neq \Phi(s)$ and $g \in G$ such that there is an edge in $\mathcal{N}_S$ between s and $(g; c, d)$ and between t and $(g; c, d)$. Hence a shortest path between s and t has length at most 2.

(2) Assume $K = \varnothing$. Then, by Theorem 3.7.(4), every element of S belongs to a stem. By Theorem 3.7.(6), we also get that S has only one stem. Part (1) thus yields that S is connected and a shortest path between any two distinct elements $s, t \in S$ has length at most 2.

(3) Assume that S does not have any connections. Suppose $S^{(i)}$ is a stem, $s \in S^{(i)}$ and $t \notin S^{(i)}$ and assume there is an edge in $\mathcal{N}_S$ between s and t. Corollary 3.9 implies that there exists a stem $S^{(k)}$ such that $s, t \in S^{(k)}$. Let p be such that $s \in S_p \backslash S_{p+1}$ and S_p / S_{p+1} is a principal factor. Then $S_p \backslash S_{p+1}$ is a connection between $S^{(i)}$ and $S^{(k)}$. This contradicts with the assumption that S does not have any connections. It follows that any connected component with more than one element is contained in a stem. Because of part (1) we actually get that such a connected component is a stem. Furthermore, by Theorem 3.7.(5), a stem is a subsemigroup.

(4) Suppose $S_k \backslash S_{k+1}$ is a connection between two stems $S^{(i)}$ and $S^{(j)}$. Let $s, t \in S^{(i)} \cup S^{(j)}$ and let $x \in S_k \backslash S_{k+1}$. Since s and x belong to the same stem, by part (1) they are connected by a path of length at most 2. By the same reson t and x are connected by a path of length at most 2. Therefore the result follows.

□

Corollary 3.11. *Let S be a pseudo-nilpotent finite semigroup. The following properties hold.*

1. *Every connected component of $\mathcal{N}_S$ that has more than one element is a union of some stems.*
2. *If C is a connected component of S then $C \cup K$ is a semigroup.*
3. *If $C_1, \ldots, C_n$ are the connected components of S with more than one element then $S/K = C_1^\theta \cup \cdots \cup C_n^\theta$, a 0-disjoint union.*

Proof. This follows at once from Theorem 3.7 and Corollary 3.10. □

So we have shown that every non-isolated connected component of a pseudo-nilpotent finite semigroup S is a union of stems, say $S_1, \ldots, S_n$. Hence, every S_i has a connection with S_j for some $i \neq j$. However, S_i is not necessarily connected with every S_j. We give an example. For this we recall from [7] that the non-commuting graph $\mathcal{M}_S$ of a semigroup S is the graph whose vertices are the elements of S and in which there is an edge between two distinct vertices x and y if these elements do not commute. By [7, Lemma 3.5], if S is a band, then $\mathcal{N}_S = \mathcal{M}_S$.

Let $X_0, X_1, \ldots, X_n$ be semigroups ($n \geq 1$) such that $X_i = \{a_i, b_i, f_i, c_i, d_i, \theta\}$, for $0 \leq i \leq n$ and X_i has Cayley table

	θ	a_i	b_i	f_i	c_i	d_i
θ	θ	θ	θ	θ	θ	θ
a_i	θ	a_i	b_i	b_i	θ	θ
b_i	θ	a_i	b_i	b_i	θ	θ
f_i	θ	a_i	b_i	f_i	c_i	d_i
c_i	θ	θ	θ	d_i	c_i	d_i
d_i	θ	θ	θ	d_i	c_i	d_i

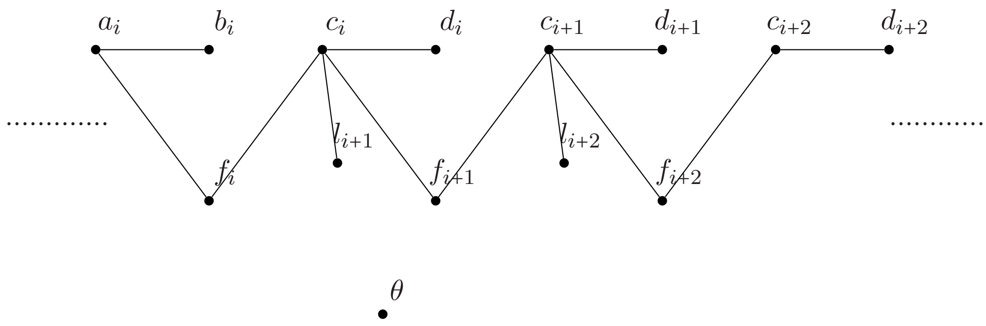

FIGURE 1.

Note that the semigroups X_i are isomorphic to the semigroup T given after Definition 3.6. Furthermore, $X_i \cap X_{i+2} = \{\theta\}$ for $0 \le i \le n-2$ and $a_{i+1} = c_i$, $b_{i+1} = d_i$, $X_i \cap X_{i+1} = \{a_{i+1}, b_{i+1}, \theta\}$ for $0 \le i \le n-1$.

We now define the semigroup

$$S = \bigcup_{0 \le i \le n} X_i \bigcup \{l_1, \dots, l_n\}$$

(where $l_1, \dots, l_n$ are the distinct elements not belonging to $\bigcup_{0 \le i \le n} X_i$) with multiplication such that each X_i is a subsemigroup and such that $X_i X_{i+2} = X_{i+2} X_i = \{\theta\}$ for $0 \le i \le n-2$. Furthermore, for $0 \le i \le n-1$,

	f_i	f_{i+1}	c_i	d_i	l_{i+1}
f_i	f_i	l_{i+1}	c_i	d_i	l_{i+1}
f_{i+1}	l_{i+1}	f_{i+1}	c_i	d_i	l_{i+1}
c_i	d_i	d_i	c_i	d_i	d_i
d_i	d_i	d_i	c_i	d_i	d_i
l_{i+1}	l_{i+1}	l_{i+1}	c_i	d_i	l_{i+1}

and $\{a_i, b_i\} f_{i+1} = f_{i+1}\{a_i, b_i\} = \{c_{i+1}, d_{i+1}\} f_i = f_i \{c_{i+1}, d_{i+1}\} = \theta$ and, for $1 \le i \le n$, we also have $l_i x = x l_i = \theta$ for $x \in S \setminus \{l_i, f_{i-1}, f_i, c_{i-1}, d_{i-1}\}$. We claim that S is pseudo-nilpotent. We prove this by contradiction. So assume that there exist distinct elements $x, y \in S$, elements $w_1, \dots, w_m \in S^1$, an ideal I of $\langle x, y, w_1, \dots, w_m \rangle$ such that

$$\lambda_t(x, y, w_1, w_2, \dots, w_t) = \lambda_m(x, y, w_1, w_2, \dots, w_m),$$

$$\rho_t(x, y, w_1, w_2, \dots, w_t) = \rho_m(x, y, w_1, w_2, \dots, w_m),$$

$\lambda_m \ne \rho_m$, $\lambda_m, \rho_m \notin I$ and $\langle \lambda_i, \rho_i \rangle$ is nilpotent in $\langle x, y, w_1, \dots, w_m \rangle / I$ for some $0 \le i \le m$. Because the subsemigroups $Y_k = \{\theta, a_k, b_k\}$ are ideals of S for $0 \le k \le n$, if $\{\lambda_i, \rho_i, w_{i+1}\} \cap Y_k \ne \varnothing$ for some $0 \le k \le n$, then $\{\lambda_{i+1}, \rho_{i+1}\} \subseteq Y_k$. Since $\lambda_m \ne \rho_m$, $\{\lambda_{i+1}, \rho_{i+1}\} = \{a_k, b_k\}$ and thus $a_k, b_k \in S \setminus I$. Also

since $\lambda_{i+1}, \rho_{i+1} \in \langle x, y, w_1, \ldots, w_m \rangle$, we have that $a_k, b_k \in \langle x, y, w_1, \ldots, w_m \rangle$. Therefore there is an edge between (a_k, b_k) in $\mathcal{N}_{\langle x,y,w_1,\ldots,w_m\rangle/I}$.

We claim that $a_k \notin S(S\backslash\{a_k\})$. Indeed, suppose the contrary, i.e. assume $\alpha\beta = a_k$ with $\alpha \in S$ and $\beta \in (S\backslash\{a_k\})$. Then $\alpha\beta\beta = a_k\beta$. Since S is band, $a_k = \alpha\beta = a_k\beta$. Now as $a_k(S\backslash\{a_k\}) = \{\theta, b_k\}$ we get that $\beta = a_k$, a contradiction. This proves the claim. Now as $a_k \in \{\lambda_i w_{i+1}\rho_i, \rho_i w_{i+1}\lambda_i\}$ and $a_k \notin S(S\backslash\{a_k\})$, $a_k \in \{\lambda_i, \rho_i\}$. Suppose that $\lambda_i = a_k$. Since there is no edge between λ_i and ρ_i, $\rho_i \in S\backslash\{a_k, b_k, f_k, f_{k-1}, l_k\}$, because if $\rho_i \in \{b_k, f_{k-1}, f_k, l_k\}$ then there are edges between λ_i and ρ_i in $\mathcal{N}_{\langle x,y,w_1,\ldots,w_m\rangle/I}$. Now as $a_k w_{k+1} \in Y_k$ and $Y_k(S\backslash\{a_k,\ b_k, f_k, f_{k-1}, l_k\}) = \{\theta\}$, $\theta \in \{\lambda_{i+1}, \rho_{i+1}\}$, a contradiction. Hence $\{\lambda_i, \rho_i, w_{i+1}\} \subseteq \{f_0, \cdots, f_n, l_1, \cdots, l_n\}$.

Since $\{f_0, \cdots, f_n, l_1, \cdots, l_n\}l_k = l_k\{f_0, \cdots, f_n, l_1, \cdots, l_n\} = \{\theta, l_k\}$ for every $1 \le k \le n$, $\{\lambda_i, \rho_i, w_{i+1}\} \subseteq \{f_0, \cdots, f_n\}$. Also $\{f_0, \cdots, f_{k-1}, f_{k+1}, \cdots, f_n\}f_k = f_k\{f_0, \cdots, f_{k-1}, f_{k+1}, \cdots, f_n\} = \{\theta, l_k\}$ for every $1 \le k \le n$. Thus $\{\lambda_{i+1}, \rho_{i+1}\} = \{\theta, l_k\}$. Then S is pseudo-nilpotent.

As S is a band, we have that $\mathcal{N}_S = \mathcal{M}_S$. The graph $\mathcal{N}_S$ is depicted in Figure 1. Between b_0 and d_n the shortest path has a length $4 + 2n$. Between the roots $\{a_i, b_i\}$ and $\{a_{i+1}, b_{i+1}\}$, there is a connection, but there is no connection between the roots $\{a_i, b_i\}$ and $\{a_{i+2}, b_{i+2}\}$.

We introduce a class of pseudo-nilpotent semigroups for which the connectivity between the stems is transitive.

Definition 3.12. *A pseudo-nilpotent semigroup S is said to be strong pseudo-nilpotent if it satisfies the following properties.*

- (H1) *If $\mathcal{N}_S$ has a connection $S_i\backslash S_{i+1}$ between two stems $S^{(i_1)}$ and $S^{(i_2)}$, then the pseudo-nilpotent homomorphisms Φ_1 and Φ_2 with domains $(S/S_{i_1+1})^0$ and $(S/S_{i_2+1})^0$ respectively are such that $S_i\backslash S_{i+1}$ intersects two different pseudo-nilpotent classes of Φ_1 and it also intersects two different pseudo-nilpotent classes of Φ_2.*
- (H2) *If S_i/S_{i+1} and S_j/S_{j+1} are principal factors of S with $i < j$ and there is an edge in $\mathcal{N}_S$ between some of their elements , then $S_i\backslash S_{i+1} \subset F_{S_j/S_{j+1}}(S/S_{j+1})$.*

Note that property (H1) implies that each connection intersects non-trivially the different pseudo-nilpotent classes of the pseudo-nilpotent homomorphism determined by the stem in which it is contained. Hence, Lemma 3.2.(4) easily yields that if $S_i\backslash S_{i+1}$ and $S_j\backslash S_{j+1}$ are different connections that are in a same stem, then there exists $(s,t) \in \mathcal{N}_S$ with $s \in S_i\backslash S_{i+1}, t \in S_j\backslash S_{j+1}$. Also if $i < j$, property (H2) implies $S_i\backslash S_{i+1} \subset F_{S_j/S_{j+1}}(S/S_{j+1})$.

An example of a strong pseudo-nilpotent semigroup is the semigroup $R = \{\theta, a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$ with multiplication table

	θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3
θ	θ	θ	θ	θ	θ	θ	θ	θ
a_1	θ	a_1	a_2	θ	θ	a_1	a_2	a_2
a_2	θ	a_1	a_2	θ	θ	a_1	a_2	a_2
a_3	θ	θ	θ	a_3	a_4	a_3	a_3	a_4
a_4	θ	θ	θ	a_3	a_4	a_3	a_3	a_4
b_1	θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3
b_2	θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3
b_3	θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3

With a similar proof to the one given for the example stated before Definition 3.12 one shows that R is pseudo-nilpotent. Furthermore, R has two roots $\{a_1, a_2\}$, $\{a_3, a_4\}$, with respective stems say R_1 and R_2. The set $\{b_1, b_2, b_3\}$ is a connection between R_1 and R_2. The sets $\{b_1\}$ and $\{b_2, b_3\}$ belong to different pseudo-nilpotent classes determined by the root $\{a_1, a_2\}$ and the sets $\{b_1, b_2\}$ and $\{b_3\}$ belong to different pseudo-nilpotent classes determined by the root $\{a_3, a_4\}$. Therefore R is a strong pseudo-nilpotent semigroup.

Lemma 3.13. *Let S be a strong pseudo-nilpotent finite semigroup. The following properties hold.*

(1) *If there is a connection between the stems $S^{(i_1)}$ and $S^{(i_2)}$ and also between the stems $S^{(i_2)}$ and $S^{(i_3)}$, then there is a connection between the stems $S^{(i_1)}$ and $S^{(i_3)}$.*
(2) *If there is no connection between two stems $S^{(i)}$ and $S^{(j)}$ then these stems belong to different connected components of $\mathcal{N}_S$.*

Proof. (1) Suppose $S_{k_1} \backslash S_{k_1+1}$ is a connection between the stems $S^{(i_1)}$ and $S^{(i_2)}$ and $S_{k_2} \backslash S_{k_2+1}$ is a connection between the stems $S^{(i_2)}$ and $S^{(i_3)}$. Suppose that $k_1 < k_2$. Since S is a strong pseudo-nilpotent semigroup, $S_{k_1} \backslash S_{k_1+1} \subseteq F_{S_{k_2}/S_{k_2+1}}(S/S_{k_2+1})$ and thus $S_{k_1} \backslash S_{k_1+1} \subseteq S^{(k_2)}$. Since $S_{k_2} \backslash S_{k_2+1} \subseteq S^{(i_3)}$, we obtain from Theorem 3.7.(2) that $S_{k_1} \backslash S_{k_1+1} \subseteq S^{(i_3)}$. Therefore $S_{k_1} \backslash S_{k_1+1}$ is a connection between the stems $S^{(i_1)}$ and $S^{(i_3)}$.

(2) Let $S^{(i)}$ and $S^{(j)}$ be two different stems. Suppose that $x \in S^{(i)}$, $y \in S^{(j)}$ and that $x = z_0, z_1, \dots, z_n, z_{n+1} = y$ is a path between x and y. Because of Theorem 3.7.(4), for $0 \le i \le n$, we get that there exist stems $S^{(n_{z_i})}$ and subsets $S_{m_{z_i}} \backslash S_{m_{z_i}+1}$ of $S^{(n_{z_i})}$ such that $z_i \in S_{m_{z_i}} \backslash S_{m_{z_i}+1}$ for a principal factor $S_{m_{z_i}}/S_{m_{z_i}+1}$.

Suppose that $m_{z_i} < m_{z_{i+1}}$. Since S is a strong pseudo-nilpotent semigroup and there is an edge in $\mathcal{N}_S$ between z_i and z_{i+1}, we have $S_{m_{z_i}} \backslash S_{m_{z_i}+1} \subseteq F_{S_{m_{z_{i+1}}}/S_{m_{z_{i+1}}+1}}(S/S_{m_{z_{i+1}}+1})$. Hence $S_{m_{z_i}} \backslash S_{m_{z_i}+1}$ is a connection between the stems $S^{(n_{z_i})}$ and $S^{(n_{z_i}+1)}$. Similarly we have this result for $m_{z_i} > m_{z_{i+1}}$.

Therefore by part (1), there is a connection between $S^{(i)}$ and $S^{(j)}$. This contradicts with there is no connection between them. □

Note that if S is a pseudo-nilpotent finite semigroup that is not strong pseudo-nilpotent then in general Lemma 3.13 does not hold. For example, the pseudo-nilpotent semigroup with graph as depicted in Figure 1, does not satisfy property (H1) and it satisies neither (1) nor (2) of Lemma 3.13. An example of a pseudo-nilpotent finite semigroup that does not satisfy property (H2) and it satisfies neither (1) nor (2) of Lemma 3.13, is the semigroup $T = \{\theta, a_1, a_2, a_3, b_1, b_2, b_3, f_1, f_2, f_3, f_4\}$ with multiplication table

	θ	a_1	b_1	a_2	b_2	a_3	b_3	f_1	f_2	f_3	f_4
θ	θ	θ	θ	θ	θ	θ	θ	θ	θ	θ	θ
a_1	θ	a_1	b_1	θ	θ	θ	θ	b_1	a_1	θ	θ
b_1	θ	a_1	b_1	θ	θ	θ	θ	b_1	a_1	θ	θ
a_2	θ	θ	θ	a_2	b_2	θ	θ	b_2	a_2	b_2	a_2
b_2	θ	θ	θ	a_2	b_2	θ	θ	b_2	a_2	b_2	a_2
a_3	θ	θ	θ	θ	θ	a_3	b_3	θ	θ	b_3	a_3
b_3	θ	θ	θ	θ	θ	a_3	b_3	θ	θ	b_3	a_3
f_1	θ	a_1	b_1	a_2	b_2	θ	θ	f_1	f_2	b_2	a_2
f_2	θ	a_1	b_1	a_2	b_2	θ	θ	f_1	f_2	b_2	a_2
f_3	θ	θ	θ	a_2	b_2	a_3	b_3	b_2	a_2	f_3	f_4
f_4	θ	θ	θ	a_2	b_2	a_3	b_3	b_2	a_2	f_3	f_4

We leave it to the reader to verify that T is pseudo-nilpotent. Further, the semigroup T has three roots $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$, with respective stems say T_1, T_2 and T_3. The set $\{f_1, f_2\}$ is a connection between T_1 and T_2 and the set $\{f_3, f_4\}$ is a connection between T_2 and T_3, but there is no connection between T_1 and T_3. As

$$\{f_3, f_4\} \subset F'_{\{a_1,b_1,a_2,b_2,a_3,b_3,f_1,f_2,\theta\}/\{a_1,b_1,a_2,b_2,a_3,b_3,\theta\}}(T/\{a_1, b_1, a_2, b_2, a_3, b_3, \theta\}),$$

T is not a strong pseudo-nilpotent semigroup.

Corollary 3.14. *Let S be a strong pseudo-nilpotent finite semigroup with K the ideal of isolated vertices. Two elements x and y of $S\backslash K$ are in the same connected component of $\mathcal{N}_S$ if and only if there exists an element $z \in S$ such that $xz \notin K$ and $zy \notin K$.*

Proof. Suppose x and y are in $S\backslash K$. Assume x and y are in a same stem, say $S^{(h)}$. Because of Theorem 3.7.(5), $S^{(h)}$ is a subsemigroup of S and because of Corollary 3.10.(1), $S^{(h)}$ is connected. Hence, by Theorem 3.7.(5), $xy, y^2 \notin K$ and x and y are in the same connected component.

Now assume that x and y are not in a same stem. Because of Theorem 3.7.(4), there exist stems $S^{(i)}, S^{(j)}$ such that $x \in S^{(i)}$ and $y \in S^{(j)}$. We need to deal with two cases: (i) between $S^{(i)}$ and $S^{(j)}$, there is a connection, say $S_k\backslash S_{k+1}$, (ii) between $S^{(i)}$ and $S^{(j)}$ there is no connection.

(i) Let $z \in S_k \backslash S_{k+1} \subseteq S^{(i)} \cap S^{(j)}$. Since $S^{(i)}$ and $S^{(j)}$ are semigroups we get that $xz \in S^{(i)}$ and $zy \in S^{(j)}$ and thus $xz \notin K$ and $zy \notin K$. Corollary 3.10.(4) implies that x and y are in the same connected component.

(ii) Since, by assumption, there is no connection between $S^{(i)}$ and $S^{(j)}$, Lemma 3.13.(2) yields that x and y are in different connected components. Now let $z \in S$ and assume $xz, zy \notin K$. Then, by Theorem 3.7.(6), there exists stems $S^{(k_1)}$ and $S^{(k_2)}$ such that $x, z \in S^{(k_1)}$ and $z, y \in S^{(k_2)}$. Corollary 3.10.(4) implies that x and y are in the same connected component, a contradiction.

The result follows. □

Definition 3.15. *Let S be a pseudo-nilpotent finite semigroup and let K be its largest nilpotent ideal. For $a \in S$ put $K_a = \{s \in S \mid as \notin K\}$. Because of Theorem 3.7.(6), $K_a = \{s \in S \mid sa \notin K\}$.*

Corollary 3.16. *Let S be a strong pseudo-nilpotent finite semigroup. The connected components with more than one element are the maximal elements in the set $\{K_a \mid a \in S \backslash K\}$.*

Proof. By Theorem 3.7, if a connected component of $a \in S$ has more than one element then $a \in S \backslash K$. Furthermore, $a \in K_a$ for any $a \in S \backslash K$. Let $x \in S \backslash K$ be such that K_x is a maximal element in the set $\{K_a \mid a \in S \backslash K\}$. We need to prove that K_x is a connected component.

First we notice that K_x is connected. Indeed, if $y \in K_x$, then $xy \notin K$. Hence, again by Theorem 3.7, x and y are in a same stem. Since every stem is connected by Corollary 3.10, there is a path between x and y. Hence all the element of K_x are in the same connected component.

Second, suppose that $z \in S$ is in the connected component containing x. It remains to be shown that $z \in K_x$. To do so, we first notice from Corollary 3.14 that there exists $v \in S$ such that $xv \notin K$ and $vz \notin K$. Because of Theorem 3.7.(6), there exist stems $S^{(i_1)}$ and $S^{(i_2)}$ such that $x, v \in S^{(i_1)}$, $z, v \in S^{(i_2)}$. Assume $x \in S_{n_x} \backslash S_{n_x+1}$, $z \in S_{n_z} \backslash S_{n_z+1}$ and $v \in S_{n_v} \backslash S_{n_v+1}$, where $T_1 = S_{n_x}/S_{n_x+1}$, $T_2 = S_{n_z}/S_{n_z+1}$ and $T_3 = S_{n_v}/S_{n_v+1}$ are principal factors of S. As S is strong pseudo-nilpotent and T_3 is a connection between the stems $S^{(i_1)}$ and $S^{(i_2)}$, T_3 intersects non-trivially different pseudo-nilpotent classes of the pseudo-nilpotent homomorphisms determined by these stems. Hence by Lemma 3.2.(4), there is an edge between some non-zero elements of T_1 and T_3 and there also is an edge between some non-zero elements of T_2 and T_3 in $\mathcal{N}_S$. Consequently, either $(T_1 \backslash \{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$ or $(T_3 \backslash \{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$, and either $(T_2 \backslash \{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$ or $(T_3 \backslash \{\theta\}) \subseteq F_{T_2}(S/S_{n_z+1})$. We therefore need to deal with four cases.

(Case 1) $(T_1 \backslash \{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$ and $(T_3 \backslash \{\theta\}) \subseteq F_{T_2}(S/S_{n_z+1})$.
By Theorem 3.7.(2), we get that $(T_1 \backslash \{\theta\}) \subseteq F_{T_2}(S/S_{n_z+1})$. Hence, $(T_1 \backslash \{\theta\})(T_2 \backslash \{\theta\}) \subseteq S^{(i_2)}$. So, by Theorem 3.7.(6), $T_1 \backslash \{\theta\}$ and $T_2 \backslash \{\theta\}$ are in a same stem. Consequently, by Theorem 3.7.(5), $xz \notin K$ and thus $z \in K_x$.

(Case 2) $(T_1\backslash\{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$ and $(T_2\backslash\{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$.
In this case, by Theorem 3.7.(2), $x, z \in (T_1\backslash\{\theta\}) \cup (T_2\backslash\{\theta\}) \subseteq S^{(i_2)}$. Hence, by Theorem 3.7.(5), $xz \notin K$. So again $z \in K_x$.

(Case 3) $(T_3\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$ and $(T_2\backslash\{\theta\}) \subseteq F_{T_3}(S/S_{n_v+1})$.
As in (Case 1) one obtains that $z \in K_x$.

(Case 4) $(T_3\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$ and $(T_3\backslash\{\theta\}) \subseteq F_{T_2}(S/S_{n_z+1})$.
Clearly $x, z \in K_v$. Assume $y \in K_x$ (and thus, in particular, $y \notin K$) and $y \in S_{n_y}\backslash S_{n_y+1}$ for some principal factor $T_4 = S_{n_y}/S_{n_y+1}$.

We claim that $y \in K_v$. Let $S^{(i_3)}$ be a stem such that $x, y \in S^{(i_3)}$. If $S^{(i_3)} = S^{(i_1)}$, then by Theorem 3.7.(5), $yv \notin K$ and thus $y \in K_v$. Suppose that $S^{(i_3)} \neq S^{(i_1)}$. Then T_1 is a connection between $S^{(i_1)}$ and $S^{(i_3)}$. Since S is strong pseudo-nilpotent, we know that either $(T_4\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$ or $(T_1\backslash\{\theta\}) \subseteq F_{T_4}(S/S_{n_y+1})$. If $(T_4\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$ then since $(T_3\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$, as in Case 2 we obtain that $vy \notin K$ and thus $y \in K_v$. If $(T_1\backslash\{\theta\}) \subseteq F_{T_4}(S/S_{n_y+1})$ then, since $(T_3\backslash\{\theta\}) \subseteq F_{T_1}(S/S_{n_x+1})$, as in Case 1 we obtain that $vy \notin K$ and thus $y \in K_v$. This finishes the proof of the claim.

So we have proved that $K_x \subseteq K_v$. As, by assumption, K_x is a maximal element in the set $\{K_a \mid a \in S\backslash K\}$, we get that $K_x = K_v$. Since $z \in K_v$ we thus obtain that $z \in K_x$, as desired. $\square$

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